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DEVELOPMENT OF METHODS FOR DETERMINING DEMAND-LIMITING SETPOINT TRAJECTORIES IN COMMERCIAL BUILDINGS USING SHORT-TERM DATA ANALYSIS

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ABSTRACT

This paper presents simple approaches for estimating building zone temperature set-point variations that minimize peak cooling demand during critical demand periods. Three different methods were developed that are termed the semi-analytical (SA), exponential setpoint equation-based semi-analytical (ESA), and load weighted-averaging (WA) methods. The three methods are different in terms of requirements for input data and ability to reduce peak demand. The SA and ESA methods employ simple inverse building models trained with short-term data and use analytical solutions from the models to determine setpoint trajectories. The WA method is a data-based method in which an optimal weighing factor is found that minimizes a weighted-average of two loads. In addition to determining setpoint trajectories, the methods provide estimates of peak load reduction. A companion paper (Lee and Braun 2006b) presents evaluations of the peak load reduction potential associated with implementation of these methods.

INTRODUCTION

There have been a number of simulation and experimental studies that have demonstrated significant potential for reducing peak cooling demand using building thermal mass through control of zone temperatures (e.g., Braun 1990, Braun et al. 2001, Lee and Braun 2004, and Xu et al. 2006). However, there has been very little work on the develoment of practical control methods for minimizing peak demand. Lee and Braun (2004) developed a model-based demand-limiting method that relies on a detailed inverse model. The method was trained using data from the Energy Resource Station building that houses the Iowa Energy Center and validated experimentally by Lee and Braun (2006a). The test results showed 30% reductions in peak cooling loads with setpoint adjustments from 70 to 76°F for a 5-hour demand-limiting. These results are consistent with the simulation results of Lee and Braun (2004) for this facility. The model-based method described by Lee and Braun (2004, 2006a) employs a detailed inverse model that requires a lot of training data and measurements that are not typically available for most buildings (e.g., solar radiation). There is a need for simpler approaches.

Relatively little work has been done in developing simple demand-limiting approaches for adjusting zone temperature setpoints that give near-optimal performance. A simple analytic method that uses a first-order model for the whole building was studied by Rabl and Norford (1991). Ambient temperature and solar radiation were eliminated by taking the difference between modeling equations for two controls, i.e. conventional and setpoint adjustment control. Peak reduction potential was calculated for a building with known building time constants for ‘subcooling’ and ‘warm-up’ periods by assuming energy consumption was constant during the on-peak period. More recently, Braun and Lee (2006) developed a simple setpoint equation for demand-limiting from a simple indoor building model. In this approach, effective time constants were determined with a trial-and-error method. Peak load reduction was evaluated through simulation for some representative small commercial buildings. As a fraction of the baseline peak under conventional control, the demand reduction ranged from about 30% to 100% depending on the climate.

The current paper builds on previous work (Lee and Braun 2004, 2006a) and develops three practical methods for determining demand-limiting setpoint trajectories. The methods differ in terms of implementation requirements and performance. A detailed evaluation of the three approaches is presented in a companion paper (Lee and Braun 2006b).

DEMAND-LIMITING CONTROL USING BUILDING THERMAL MASS

Figure 1 depicts temperature setpoint changes for demand-limiting control methods that utilize building thermal mass during a critical peak period in the afternoon. In order to precool the structure, building temperature setpoints are set at a lower bound of comfort until the demand-limiting period begins. During the demand-limiting period, the setpoints are adjusted between lower and upper bounds of comfort following a trajectory that minimizes the peak load requirement. Limited test results from Lee and Braun (2006a) indicate that occupant comfort is not significantly affected when zone temperatures are maintained at 70°F during morning hours and then raised to 78°F during afternoon. Variation of the setpoints controls the rate of heat gains from the interior surfaces and has a profound effect on the shape of the load time variation. Simple methods for setpoint adjustment include 'linear-rise' and 'step-up' trajectories that are depicted in Figure 1. However, these methods have been shown by Lee and Braun (2006a) not to be optimal for minimizing peak demand. In this paper, three demand-limiting methods for estimating optimal setpoint trajectories are developed. All three methods require short-term load data obtained from buildings during afternoon periods that are characteristic of periods where the demand-limiting will be applied.

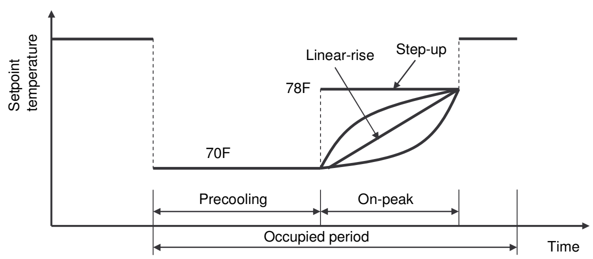

Figure 1 Example demand-limiting building setpoint temperature controls

DEMAND-LIMITING METHODS

Semi-Analytical (SA) Method

The SA method determines an analytical expression for demand-limiting setpoint from a simple building model that characterizes thermal interactions between the interior space and a "shallow" interior mass. A schematic diagram of the method is illustrated in Figure 2. Actual cooling load data under conventional control are used for estimating parameters associated with a simple building model. The parameters are then used within an analytic expression for the demand-limiting setpoint trajectory.

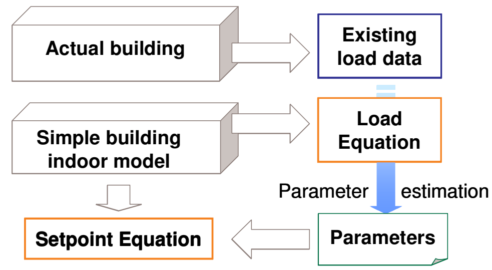

Figure 2 Schematic illustration of SA method

Figure 3 depicts the simple interior mass building model that is used for the SA method to describe the thermal behavior of a building over the demand-limiting period. In this figure, T_a is outdoor air temperature, T_z is zone air temperature, C_{ms} is thermal capacitance of the shallow mass, R_d is thermal resistance between the shallow and deep mass, R_s is resistance between the zone air and shallow mass, T_{md} is temperature of the deep mass, T_{ms} is temperature of the shallow mass, $Q_{g,c}$ is convective heat gain to the zone air, $Q_{g,r}$ is radiative heat transfer to the shallow mass surfaces due to internal sources and solar radiation transmitted through windows, and Q_z is zone sensible cooling load. The building network model characterizes the sensible cooling requirement assuming a deep mass temperature is nearly constant over the relatively short demand-limiting period. Radiative heat gain involving transmitted solar radiation into the building space acts on the shallow mass node and convective heat gain occurs to a zone temperature node. Short-term coupling of the zone and outdoor air occurs due to combined effects of conduction heat transfer within the window and convection due to infiltration and ventilation.

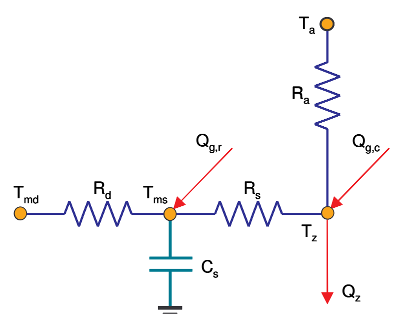

Figure 3 Simple indoor mass building model for SA method

From the simple indoor mass building model, two analytic expressions are derived: 1) the cooling load requirement under conventional control and 2) a zone setpoint temperature trajectory for minimizing the peak cooling load. The governing differential equations for the simple indoor building model are:

$$C_{ms}\frac{dT_{ms}}{dt}=\frac{T_{md}-T_{ms}}{R_d}+\frac{T_z-T_{ms}}{R_s}+Q_{g,r} \quad (1)$$

$$0=\frac{T_{ms}-T_z}{R_s}+\frac{T_a-T_z}{R_a}+Q_{g,c}-Q_z \text{ for } t<0\leq t_{dl}. \quad (2)$$

In developing the SA method, it was assumed that during the demand-limiting period the outdoor temperature, T_a, can be expressed as a quadratic polynomial function of time and the radiative heat gain, $Q_{g,r}$, can be represented using a cubic polynomial variation with time. Radiative heat gain is not directly measured and therefore it was assumed that its time variation is related to the variation in cooling load through a constant time lag, g_t, and a constant shift factor, g. Equations for the radiative heat gain and outdoor temperature are given in Appendix A.

Parameters associated with equations (1) and (2) are determined using data for the building operating under conventional control with fixed zone setpoint temperatures during the demand-limiting period. Under these conditions, the shallow mass temperature, T_{ms}, in equations (1) and (2) is eliminated and the resulting equation is rearranged to give a first-order differential equation for zone sensible cooling load with fixed zone temperature, $Q_{z,cc}$ (see Appendix B). The differential equation is solved using an initial condition of $Q_{z,cc}(0)=Q_{z,cc,i}$. Appendix B gives the development and resulting analytical expression for the cooling load. The generic dependence of the cooling load on time and building-specific parameters is expressed as

$$Q_{z,cc}=f(t:C_s,R_d,R_s,R_a,g,g_t,T_{md,cc},Q_{g,c}) \quad (3)$$

where $T_{md,cc}$ is the deep mass temperature associated with conventional control. The building parameters within the cooling load equation (3) are estimated using non-linear regression applied to cooling load data obtained for demand-limiting periods where zone temperature is constant.

In order to determine the demand-limiting setpont trajectory, it is assumed that a constant cooling load is optimal for the demand-limiting period. With $Q_z = Q_{z,dl}$ = constant, the term T_{ms} from equations (1) and (2) is eliminated and the resulting equation is rearranged for T_z to yield a first-order differential equation for zone temperature. The differential equation is solved to give an analytical expression for zone temperature using an initial condition of $T_{z,dl}(0)=T_{z,i}$ (e.g., a precooling temperature at the lower bound of acceptable comfort). The solution is termed the 'open-ended' demand-limiting setpoint equation to signify that the zone temperature during the demand-limiting period is not constrained. The development and resulting expression are given in Appendix C, whereas the functional dependence is expressed as

$$T_{z,dl}=f(t:C_{ms},R_d,R_s,R_a,g,g_t,T_{md,dl},Q_{g,c},Q_{z,dl})\,. \quad (4)$$

A closed-ended form of the demand-limiting equation is obtained by applying a constraint for the setpoint at the end of the demand-limiting period (e.g., the upper limit for acceptable comfort) such that $T_{z,dl}(t_{dl})=T_{z,f}$. The application of this constraint allows elimination of the deep mass temperature, convective gains and demand-limiting cooling rate. The development and resulting expression are given in Appendix D and the functional dependence is described by

$$T_{z,dl}=f(t:C_s,R_d,R_s,R_a,g,g_t)\,. \quad (5)$$

The closed-ended setpoint equation provides a simple means for estimating a zone temperature setpoint variation during the demand-limiting period that results in a constant cooling requirement and is bounded between minimum and maximum limits of comfort.

The simple building model can also be used to estimate peak load reduction associated with the demand-limiting strategy. The conventional peak load is estimated directly with equation (3), whereas the peak load for demand-limiting control is obtained using equation (4) by numerically determining the cooling rate that gives a final zone temperature at the end of the demand-limiting period that is equal to the upper comfort constraint. In this case, the deep mass temperature under conventional control is used with equation (4) even though the temperature under demand-limiting control is thought to be lower than that under conventional control. This assumption may lead to overestimates of peak load under demand-limiting control.

Exponential Setpoint Equation-Based Semi-Analytical (ESA) Method

A simple exponential equation for demand-limiting control was derived by Braun and Lee (2006) assuming that all driving input conditions are constant during the demand-limiting period.

$$\frac{T_{z,dl}-T_{z,i}}{T_{z,f}-T_{z,i}}=\frac{1-\exp(-t/\tau)}{1-\exp(-t_{dl}/\tau)} \text{ for } 0\leq t\leq t_{dl} \quad (6)$$

where $T_{z,dl}$ is the setpoint temperature, $T_{z,i}$ is the initial temperature at the start of demand-limiting period (e.g., 70°F), $T_{z,f}$ is the temperature at the end of the demand-limiting period (e.g., 78°F), t is time measured from the start of the demand-limiting period, t_{dl} is the length of the demand-limiting period, and τ is an effective time constant for the setpoint trajectory. This simple exponential has been shown by Braun and Lee (2006) to be very effective in peak demand reduction with a proper effective time constant. Simulation results presented by Braun and Lee (2006) were obtained for some prototype buildings by estimating time constants using a trial-and-error method. However, it is desireable to have a general methodology for estimating time constants from short-term measurements.

The ESA method produces a time constant for equation (6) and is illustrated in Figure 4. The method requires cooling load data for two different control strategies implemented on two different days, Q_1 and Q_2. One of the strategies should be conventional control and the other a simple demand-limiting strategy, such as a 'linear-rise' setpoint strategy. The two sets of load are used to estimate parameters for a simple building model. The parameters are then used to find an effective time constant that minimizes peak demand.

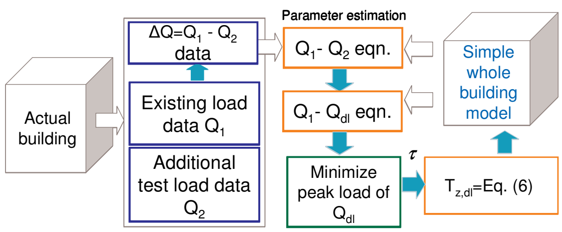

Figure 4 Overview of ESA method

The parameter estimation is applied to a simple whole building mass model that is depicted in Figure 5. In this representation, the building mass node is at a temperature of T_m and characterizes the entire effective building mass. Solar radiation, $Q_{g,s}$, and internal radiative heat gain, $Q_{g,r}$, both act on the building mass node. The building mass is also coupled directly to the outdoor air, zone air, and ground. A massless zone air node is connected to the building mass node and outdoor air. The zone air coupling to the outdoor air represents the thermal resistance through windows and convection resulting from infiltration and ventilation. There is also convective heat gain to the zone air from lighting and equipment within the interior spaces.

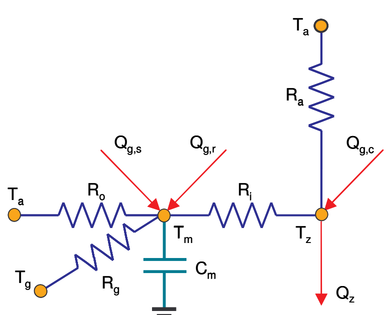

Figure 5 Simple whole building model for ESA method

The governing differential equations for this simple whole building model can be written as:

$$C_m \frac{dT_m}{dt} = \frac{T_a - T_m}{R_o} + \frac{T_g - T_m}{R_g} + \frac{T_z - T_m}{R_i} + Q_{g,s} + Q_{g,r} \tag{7}$$

$$0 = \frac{T_m - T_z}{R_i} + \frac{T_a - T_z}{R_a} + Q_{g,c} - Q_z \quad \text{for } t<0\leq t_{dl} \tag{8}$$

where C_m is thermal capacitance of the effective building mass, R_o is thermal resistance between the outdoor air and effective building mass, R_i is thermal resistance between the zone air and effective building mass, T_m is temperature of the effective building mass, T_a is temperature of the outdoor air, $Q_{g,c}$ is convective heat gain to the zone air, $Q_{g,s}$ is the solar radiation on the exterior building walls, $Q_{g,r}$ is radiative heat transfer to interior building mass surfaces due to internal sources and solar transmitted through windows, and Q_z is the zone sensible cooling load. Equation (6) arises from the solution to these differential equations with an assumption of constant driving conditions.

In applying the ESA method, it is assumed that all driving conditions, including outdoor temperature, solar radiation, radiative heat gain, and internal convective heat gain, are similar for different days. This assumption eliminates the need to have measurements of actual driving conditions in determining the effective time constant for demand-limiting control. Now consider two different control strategies that employ setpoint trajectories produced with equation (6) for time constants τ_1 and τ_2. Equations (7) and (8) apply for each strategy and a set of equations involving differences in state variables are obtained as

$$C_m \frac{d(\Delta T_m)}{dt} = -\frac{\Delta T_m}{R_o} - \frac{\Delta T_g}{R_g} + \frac{\Delta T_z - \Delta T_m}{R_i} \tag{9}$$

$$0 = \frac{\Delta T_m - \Delta T_z}{R_i} - \frac{\Delta T_z}{R_a} - \Delta Q_z \tag{10}$$

where $\Delta T_m = T_{m,1} - T_{m,2}$, $\Delta T_z = T_{z,1} - T_{z,2}$, and $\Delta Q_z = Q_{z,1} - Q_{z,2}$. The terms involving ambient temperature, solar radiation, and radiative/convective heat gain have been eliminated. The ΔT_m term can also be eliminated by combining these equations and then the result can be rearranged to give a first-order differential equation for cooling load difference, ΔQ_z. Finally, this differential equation can be solved with an initial condition of $\Delta Q_z(0) = Q_{z,1}(0) - Q_{z,2}(0)$. The development and resulting expression are given in Appendix E and the functional dependence can be expressed as

$$\Delta Q_z(t) = f(t{:}\tau_1, \tau_2, C_m, R_o, R_i, R_a, R_g)\ . \tag{11}$$

As depicted in Figure 6, the ESA method involves two phases: building model parameter estimation and time constant estimation. The graphs in this figure represent setpoint temperature variations during occupied periods including precooling and demand-limiting periods. In the parameter estimation phase, the parameters of equation (11) are estimated using non-linear regression with cooling load difference data for two days having two different control strategies (e.g., conventional control (CC) and precooling with a linear-rise demand-limiting strategy (PC+Linear-rise)). Appendix E gives a special-case expression for load difference when one of the strategies is conventional control with T_z set to a constant $T_{z,cc}$. If the second strategy involves a linear rise in setpoint, then the time constant should be set to an artificially large number.

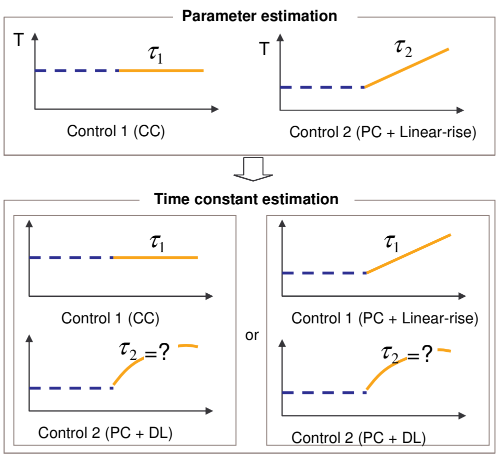

Figure 6 Two phases in ESA method

In the time constant estimation phase, one of the two training strategies (either CC or PC+Linear-rise) is utilized along with equation (11) to determine an effective time constant for a strategy that would minimize the peak cooling load (PC+DL) for a day having similar driving variables. The optimization involves minimizing the following cost function over the demand-limiting period with respect to τ_2.

$$J=\max\left(Q_{z,2}(t,\tau_2)\right)=\max\left(Q_{z,1}(t)-\Delta Q_z(t,\tau_2)\right) \text{ for } t<0\le t_{dl} \quad (12)$$

where τ_2 is the effective constant for the demand-limiting strategy (PC+DL in Figure 6), $Q_{Z,1}$ is measured load for the training strategy ('control 1' in Figure 6), and ΔQ_z is determined using equation (11). The value of J (maximum of $Q_{z,2}$) that results from this optimization is a prediction of the peak cooling demand under demand-limiting control when the ESA method is applied.

Load Weighted-Averaging (WA) Method

With the WA method, the setpoint trajectory that minimizes the peak cooling load is estimated through a weighted averaging of two control setpoint trajectories as depicted in Figure 7 (b). The two setpoint trajectories should produce load variations that intersect at some point during the demand-limiting period as shown in Figure 7 (a). The weighting factor is determined by minimizing the peak of the weighted-averaged cooling loads. The optimization problem involves minimizing

$$J=\max_{w^*}\left[wQ_{1,k}+(1-w)Q_{2,k}\right] \text{ for } k=1,..,k_{dl} \quad (13)$$

with respect to the weighting factor *w*, where $Q_{1,k}$ is the cooling load for time interval k under control 1 and $Q_{2,k}$ is the cooling load at time k under control 2.

The WA method employs the assumptions that the driving conditions (weather and internal) are similar for the two days and the cooling load at any time is a linear function of the zone temperature. With these assumptions, the zone temperature trajectory that minimizes the peak load is

$$T_{sp,dl,k}=w^*T_{sp,1,k}+(1-w^*)T_{sp,2,k} \text{ for } k=1,..,k_{dl} \quad (14)$$

where $T_{sp,1,k}$ is the setpoint temperature for time interval k with control 1, $T_{sp,2,k}$ is the setpoint temperature for control 2 at time k, $T_{sp,dl,k}$ is optimal setpoint temperature at time k, and w*is the optimal weighting factor determined by minimizing the cost function in equation (13).

The example depicted in Figure 7 shows a 'linear–rise' setpoint variation that results in a decreasing cooling load over the demand-limiting period and a 'step-up' setpoint that causes an increasing cooling load. Both setpoint variations have precooling prior to the on-peak time period. The optimal weighting factor determines the weighted averaging of these two load profiles that would minimize the peak load. When this weighting factor is applied to zone temperature profiles, a new setpoint trajectory is estimated that is between the two original setpoint trajectories.

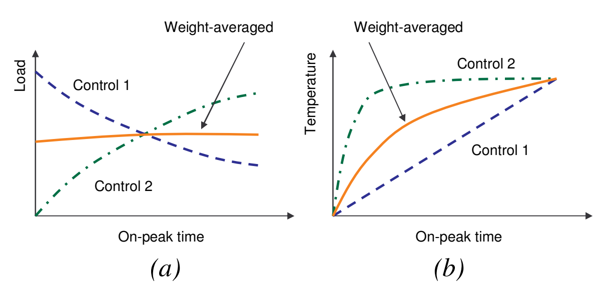

Figure 7 Schematic illustration of WA method. (a) load and (b) zone temperature

Estimating Peak Cooling Demand Reduction

The peak cooling demand associated with demand-limiting control can be estimated for each of the simple methods. The baseline peak load is determined from measurements obtained for conventional control. For the SA method, the peak cooling load for demand-limiting control can be predicted by using the closed-ended demand-limiting setpoint equation (5) and solving the open-ended setpoint equation (4) for the constant cooling rate. With the ESA method, the process of determining the effective time constant involves estimation of the peak cooling load for demand-limiting control. For the WA method, the peak cooling demand can be predicted using the load shape produced by weighting the measured load profiles for two different control strategies using the optimal weighting factor.

CONCLUSION

In this study, three practical methods that use short-term measurement data for determining demand-limiting control setpoint trajectories were developed and described. The methods also provide predictions of peak load reduction associated with implementation of the demand-limiting strategies. The SA method only requires load data for a single day under conventional control, while the ESA method requires one additional day of test data compared to the SA method. The WA method requires two additional test days with setpoint schedules that produce load profiles that bound the optimal profile. A companion paper by Lee and Braun (2006b) evaluates the peak load reduction potential associated with implementing these methods. The methods require less field data and few inputs than previous methods (Lee and Braun 2004, 2006a) and are effective in terms of peak demand reduction.

NOMENCLATURE

C = thermal capacitance
g = shift factor in approximate equation for radiative heat gain
g_t = time lag factor in approximate equation for radiative heat gain
k_{dl} = time stage at the end of demand-limiting period
Q = sensible heat transfer rate
R_a = thermal resistance between zone air and outdoor air
R_d = thermal resistance between shallow mass and deep mass
R_g = thermal resistance between ground and effective entire building mass
R_i = thermal resistance between zone air and effective building mass
R_o = thermal resistance between outdoor air and effective building mass
R_s = thermal resistance between zone air and shallow mass
T = temperature
t = time
t_{dl} = length of demand-limiting period
w = weighting factor

Subscripts

avg = average
cc = conventional control (night setup)
dl = demand-limiting control
f = final state
g = ground
g,r = radiative gain
g,c = convective gain
g,s = solar radiative gain
i = initial state
k = time stage
m = effective building mass
md = deep mass in simple building indoor mass model
ms = shallow mass in simple building indoor mass model
sp = setpoint
z = building zone air

Greeks

τ = effective time constant in simple exponential setpoint equation

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APPENDIX

Appendix A: Approximation of radiative heat gain and outdoor temperature

For purposes of learning the radiative gain profile, cooling load data under conventional control is expressed in the form of cubic polynomials as:

$$\dot{Q}_z = q_0 + q_1 t + q_2 t^2 + q_3 t^3. \quad \text{(A.1)}$$

Coefficients in the polynomial equation are obtained using regression of actual load data. It is assumed that the radiative heat gain has a similar shape as the cooling load and can be expressed as a cubic polynomial. The radiative heat gain is assumed to be related to the cooling load data through two parameters that characterize a shift in magnitude, g and a shift in time, g_t.

$$\begin{aligned} Q_{gr} &= g_0 + g_1 t + g_2 t^2 + g_3 t^3 \\ &= \left[q_0 + q_1 (t+g_t) + q_2 (t+g_t)^2 + q_3 (t+g_t)^3 \right] + g \end{aligned} \quad \text{(A.2)}$$

Then, the coefficients of g_0, g_1, g_2 and g_3 are written as

$g_0 = g + (q_0 + g_t q_1 + g_t^2 q_2 + g_t^3 q_3)$, $g_1 = q_1 + 2 g_t q_2 + 3 g_t^2 q_3$, and $g_2 = q_2 + 3 g_t q_3$ and $g_3 = q_3$.

The outdoor temperature variation for the demand-limiting period is expressed as a quadratic polynomial equation. Coefficients in the polynomial equation are obtained using regression of actual outdoor temperature data.

$$T_a(t) = T_{a0} + T_{a1} t + T_{a2} t^2 \quad \text{(A.3)}$$

Appendix B: Cooling load equation

The differential equation for cooling load under conventional control is

$$\frac{dQ_z(t)}{dt} = -\frac{1}{A_1} Q_z(t) + \frac{1}{A_2} \frac{dT_a(t)}{dt} + \frac{1}{A_3} T_a(t) + \frac{1}{A_4} Q_{gr}(t) + K_{cc} \quad \text{(B.1)}$$

where

$$K_{cc} = \frac{1}{C_{ms} R_s} \left[\frac{T_{md}}{R_d} + \frac{T_{z,cc}}{R_s} - \left(1 + \frac{R_s}{R_a}\right)\left(\frac{1}{R_d} + \frac{1}{R_s}\right) T_{z,cc} + R_s \left(\frac{1}{R_d} + \frac{1}{R_s}\right) Q_{g,c} \right] ,$$

$$\frac{1}{A_1} = \frac{1}{R_d C_{ms}} + \frac{1}{R_s C_{ms}} , \quad \frac{1}{A_2} = \frac{1}{R_a} , \quad \frac{1}{A_3} = \frac{1}{R_a}\left(\frac{1}{R_d C_{ms}} + \frac{1}{R_s C_{ms}}\right) ,$$

and $\frac{1}{A_4} = \frac{1}{R_s C_{ms}}$.

The solution of the differential equation for cooling load under conventional control with an initial condition $Q_{z,cc}(0) = Q_{z,cc,i}$ is

$$Q_{z,cc}(t) = Q_{z,cc,i} \exp\left(-\frac{t}{A_1}\right) + \frac{1}{2} F_1 \left[1 - \exp\left(-\frac{t}{A_1}\right)\right] + F_2 t^3 + F_3 t^2 + F_4 t \quad \text{(B.2)}$$

where

$$\begin{aligned} F_1 = \frac{1}{A_2 A_3 A_4} \Big[& (2 A_1 A_2 A_4) T_{a0} + 2 \left(A_1 A_3 A_4 - A_2 A_4 A_1^2 \right) T_{a1} + \\ & 4 \left(A_2 A_4 A_1^3 - A_3 A_4 A_1^2 \right) T_{a2} + 2 (A_1 A_2 A_3) g_0 - 2 \left(A_2 A_3 A_1^2 \right) g_1 + \\ & 4 \left(A_2 A_3 A_1^3 \right) g_2 - 12 \left(A_2 A_3 A_1^4 \right) g_3 + 2 (A_1 A_2 A_3 A_4) K_{dl} \Big], \end{aligned}$$

$$K_{dl} = \frac{R_a}{C_{ms}(R_a + R_s)} \left[\frac{T_{md,dl}}{R_d} - R_s \left(\frac{1}{R_d} + \frac{1}{R_s} \right) (Q_{z,dl} - Q_{g,c}) \right] , \quad F_2 = \frac{A_1}{A_4} g_3 ,$$

$$F_3 = \frac{1}{A_2 A_3 A_4} \left[(A_1 A_2 A_3) g_2 - 3 \left(A_1^2 A_2 A_3 \right) g_3 + (A_1 A_2 A_4) T_{a2} \right] , \quad \text{and}$$

$$\begin{aligned} F_4 = \frac{1}{A_2 A_3 A_4} [& (A_1 A_2 A_4) T_{a1} + 2 \left(A_1 A_3 A_4 - A_1^2 A_2 A_4 \right) T_{a2} + \\ & (A_1 A_2 A_3) g_1 - 2 \left(A_1^2 A_2 A_3 \right) g_2 + 6 \left(A_1^3 A_2 A_3 \right) g_3. \end{aligned}$$

Appendix C: Open-ended demand-limiting setpoint equation

The differential equation for the setpoint temperature under demand-limiting control is

$$\frac{dT_z(t)}{dt} = -\frac{1}{A_1} T_z(t) + \frac{1}{A_2} \frac{dT_a(t)}{dt} + \frac{1}{A_3} T_a(t) + \frac{1}{A_4} \dot{Q}_{gr}(t) + K_{dl} \quad \text{(C.1)}$$

where

$$\frac{1}{A_1} = \frac{1}{R_d C_{ms}} + \frac{1}{(R_a + R_s) C_{ms}}, \quad \frac{1}{A_2} = \frac{R_s}{R_a + R_s},$$

$$\frac{1}{A_3} = \frac{R_s}{R_a + R_s} \left(\frac{1}{R_d C_{ms}} + \frac{1}{R_s C_{ms}} \right), \text{ and } \frac{1}{A_4} = \frac{1}{C_{ms}} \left(\frac{R_a}{R_a + R_s} \right).$$

Solution of the differential equation for the demand-limiting setpoint temperature with an initial condition $T_{z,dl}(0) = T_{z,i}$ is

$$T_{z,dl}(t)=T_{z,i}\exp\left(-\frac{t}{A_1}\right)+\frac{1}{2}F_1\left[1-\exp\left(-\frac{t}{A_1}\right)\right]+F_2t^3+F_3t^2+F_4t \qquad (C.2)$$

where

$$F_1=\frac{1}{A_2A_3A_4}\Big[(2A_1A_2A_4)T_{a0}+2\left(A_1A_3A_4-A_2A_4A_1^2\right)T_{a1}+$$
$$4\left(A_2A_4A_1^3-A_3A_4A_1^2\right)T_{a2}+2(A_1A_2A_3)g_0-2\left(A_2A_3A_1^2\right)g_1+$$
$$4\left(A_2A_3A_1^3\right)g_2-12\left(A_2A_3A_1^4\right)g_3+2(A_1A_2A_3A_4)K_{dl}\Big],$$

$$F_2=(A_1/A_4)\,g_3\ ,$$

$$F_3=\frac{1}{A_2A_3A_4}\Big[(A_1A_2A_3)g_2-3\left(A_1^2A_2A_3\right)g_3+(A_1A_2A_4)T_{a2}\Big],\ \text{and}$$

$$F_4=\frac{1}{A_2A_3A_4}\Big[(A_1A_2A_4)T_{a1}+2\left(A_1A_3A_4-A_1^2A_2A_4\right)T_{a2}$$
$$+(A_1A_2A_3)g_1-2\left(A_1^2A_2A_3\right)g_2+6\left(A_1^3A_2A_3\right)g_3\Big].$$

Appendix D: Closed-ended demand-limiting setpoint equation

If the final condition $T_{z,dl}(t_{dl})=T_{z,f}$ is applied to the open-ended demand-limiting setpoint equation and the equation is re-arranged, then the following setpoint equation can be obtained. It is termed the closed-ended form of the demand-limiting setpoint equation. It should be noted that the variable F_1, which includes the terms T_{a0}, g_0, T_{dm}, $Q_{z,dl}$, and $Q_{g,c}$, does not appear in this equation.

$$\frac{T_{z,dl}(t)-T_{z,i}}{T_{z,f}-T_{z,i}}=\left[1-\exp\left(-\frac{t}{A_1}\right)\right]\Big/\left[1-\exp\left(-\frac{t_{dl}}{A_1}\right)\right]+ \qquad (D.1)$$
$$+\frac{F_4}{T_{z,f}-T_{z,i}}\left[t-t_{dl}\frac{1-\exp\left(-\frac{t}{A_1}\right)}{1-\exp\left(-\frac{t_{dl}}{A_1}\right)}\right]+\frac{F_3}{T_{z,f}-T_{z,i}}\left[t^2-t_{dl}^2\frac{1-\exp\left(-\frac{t}{A_1}\right)}{1-\exp\left(-\frac{t_{dl}}{A_1}\right)}\right]+$$
$$\frac{F_2}{T_{z,f}-T_{z,i}}\left[t^3-t_{dl}^3\frac{1-\exp\left(-\frac{t}{A_1}\right)}{1-\exp\left(-\frac{t_{dl}}{A_1}\right)}\right].$$

Appendix E: Load difference equation

The governing differential equation between load difference resulting from application of two different control strategies, 'control 1' and 'control 2', is

$$\frac{d(\Delta Q_z)}{dt}=-A_1(\Delta Q_z)+A_2(\Delta T_z)-A_3\frac{d(\Delta T_z)}{dt} \qquad (E.1)$$

where $\Delta T_z=T_{z,1}-T_{z,2}$,

$$A_1=\frac{1}{C_m}\left(\frac{1}{R_o}+\frac{1}{R_i}+\frac{1}{R_g}\right),$$

$$A_2=\frac{1}{C_mR_i}\left[\frac{1}{R_i}-\left(\frac{1}{R_o}+\frac{1}{R_i}+\frac{1}{R_g}\right)\left(\frac{R_a+R_i}{R_i}\right)\right],\ \text{and}\ A_3=\frac{R_a+R_i}{R_aR_i}.$$

If 'control 1' is conventional control (CC) with a constant setpoint and 'control 2' employs a simple exponential equation that utilizes a simple time constant, τ_2, and gives $T_{z,2}$ at each time, then the solution of equation (E.1) with an initial condition $\Delta Q_z(0)=\Delta Q_{z,0}$ is

$$\Delta Q_z(t)=\Delta Q_{z,o}\exp(-A_1t)+\frac{B_1+B_2+B_3}{B_4} \qquad (E.2)$$

where

$$\Delta T_z=T_{z,cc}-\frac{1-\exp(-t/\tau_2)}{1-\exp(-t_{dl}/\tau_2)}\left(T_{z,f}-T_{z,i}\right),\ B_1=e^{-t/\tau 2}A_1(T_{z,f}-T_{z,i})(A_3+A_2\tau_2),$$

$$B_2=e^{-A_1t}[-T_{z,f}(A_1A_3+A_2)+A_1(A_2\tau_2T_{z,i}+A_3T_{z,i}-A_2\tau_2T_{z,cc})$$
$$+A_2(T_{z,cc}+e^{-tdl/\tau 2}(1-A_1\tau_2)(T_{z,i}-T_{z,cc}))],$$

$$B_3=A_2(A_1\tau_2-1)\left(e^{-tdl/\tau_2}(T_{z,i}-T_{z,cc})+T_{z,cc}-T_{z,f}\right)A_1,$$

$$B_4=\left(1-e^{-tdl/\tau_2}\right)(A_1\tau_2-1)A_1.$$

Generally, if setpoints for 'control 1' and 'control 2' are determined from a simple exponential setpoint equation with τ_1 and τ_2, then the solution of equation (E.1) with an initial condition $\Delta Q_z(0)=\Delta Q_{z,0}$ is

$$\Delta Q_z(t)=\Delta Q_{z,o}\exp(-A_1t)+\frac{B_1+B_2+B_3+B_4}{B_5} \qquad (E.3)$$

where

$$\Delta T_z=\frac{1-\exp(-t/\tau_1)}{1-\exp(-t_{dl}/\tau_1)}\left(T_{z,f1}-T_{z,i}\right)-\frac{1-\exp(-t/\tau_2)}{1-\exp(-t_{dl}/\tau_2)}\left(T_{z,f2}-T_{z,i}\right),$$

$$B_1=e^{-t/\tau 2}A_1(1-e^{-t_{dl}/\tau 1})(T_{z,i}-T_{z,f2})(A_3+A_2\tau_2)(A_1\tau_1-1),$$

$$B_2=e^{-A_1t}\Big[T_{z,f1}-T_{z,f2}+e^{-t_{dl}/\tau 1}(T_{z,i}-T_{z,f2})(A_1\tau_1-1)+e^{-t_{dl}/\tau 2}(T_{z,f1}-T_{z,i})(A_1\tau_2-1)+$$
$$(\tau_1T_{z,f2}+T_{z,i}\tau_2-T_{z,f1}\tau_2-T_{z,i}\tau_1)A_1\Big](A_3A_1+A_2),$$

$$B_3=\{A_2e^{-t_{dl}/\tau 1}\left(T_{z,i}-T_{z,f2}\right)(A_1\tau_1-1)-e^{-t_{dl}/\tau 2}(T_{z,f1}-T_{z,i})\left[e^{-t/\tau 1}A_1(A_3+A_2\tau_1)-A_2(A_1\tau_1-1)\right]$$
$$+e^{-t/\tau 1}A_1(T_{z,f1}-T_{z,i})(A_3+A_2\tau_1)\}(A_1\tau_2-1),$$

$$B_4=(1-A_1\tau_1)(A_1\tau_2-1)\left(T_{z,f1}-T_{z,f2}\right)A_2,\ \text{and}$$

$$B_5=\left(e^{-t_{dl}/\tau_1}+e^{-t_{dl}/\tau_2}-1-e^{-t_{dl}/\tau_1-t_{dl}/\tau_2}\right)(A_1\tau_1-1)(A_1\tau_2-1)A_1$$